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G-structures of twistor type and their twistor spaces

D.V. Alekseevsky

Gen. Antonova 2, kv. 99, 117 279 Moscow, Russian Federation

M.M. Graev

Prospekt Mira 4b, kv. 36, 129 041 Moscow, Russian Federation

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The notion of a group of twistor type is defined as a linear group G whose Lie algebra has an element J with $J = -1$. The twistor space Z of a G-structure with a connection on a manifold M is constructed where G is a Lie group of twistor type. It is the total space of a bundle over M with a complex affine symmetric space G/H as a fiber. A natural almost complex structure J and a horizontal distribution H on Z are defined and studied. The conditions of integrability of J and holomorphicity of H reduce to some linear conditions on the curvature tensor of the connection. They may be considered as generalized self-dual equations. It is shown that for some Lie group G these equations are fulfilled automatically. For other groups G there are some obstacles that are described in terms of a decomposition of the curvature tensor associated with a bigradation of the appropriate Koszul–Spencer complex.

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Contents

1. Introduction

The basic idea of the twistor Programme of R. Penrose [1] is to associate with a given real manifold with geometric structure some complex manifold Z ("twistor space") and to reformulate problems of the real geometry M in terms of such a complex manifold Z. As a first step towards realizing his programme, Penrose defined the twistor space for the Minkowsky space $\mathbb{R}^{1,3}$ as the complex projective space $\mathbb{C}P^3$. The established correspondence between $\mathbb{R}^{1,3}$ and $\mathbb{C}P^3$ allowing one to interpret different problems of Special Relativity and Mathematical Physics in terms of complex geometry. This approach turned out to be very fruitful.

Generalizing the Penrose construction, Atiyah, Hitchin and Singer [2] have defined the twistor space Z of a four-dimensional oriented Riemannian (or conformal) manifold M as the total space of the projectivized spinor bundle. They have defined an almost complex structure J_Z on Z and have proved that J_Z is integrable if the Weyl tensor W of M is self-dual ($*W=W$). This construction has been generalized to quaternionic Kähler and quaternionic manifolds M^{4n} by Salamon [3–5]. If $n>1$, the almost complex structure J_Z on the twistor space is always integrable. Salamon has also defined a canonical contact structure $\mathscr{H}_Z$ on Z, which is studied in ref. [6].

The twistor space for an arbitrary Riemannian manifold (M^{2n}, g) was defined by O'Brian and Rawnsley [7] as the manifold $\mathrm{Com}(M^{2n}, g)$ of all g-orthogonal complex structures in tangent spaces T_xM, $x \in M$. It has a natural almost complex structure J_Z and a distribution $\mathscr{H}_Z$. It was shown that any J_Z-holomorphic curve in Z tangent to $\mathscr{H}_Z$ defined a harmonic mapping of a Riemannian surface into (M, g). Unfortunately, the almost complex structure J_Z is not integrable in general.

To remedy this situation, Bryant [8] proposed to define the holomorphic twistor space Z_{hol} as the maximal submanifold of $\mathrm{Com}(M, g)$ on which J_Z induces an integrable complex structure and $\mathscr{H}_Z$ defines a holomorphic distribution. Bryant classified this kind of holomorphic twistor space for all Riemannian symmetric spaces. This gives a powerful method to construct harmonic mappings into symmetric manifolds, see refs. [9,10].

A general construction of the twistor space Z with an almost complex structure J_Z for an arbitrary G-structure has been proposed by Bérard-Bergery and Ochiai [11]. They give the conditions for integrability of the almost complex structure and apply it to some classical G-structures.

In this paper we define the notion of a group of twistor type as a linear group $G \subset \mathrm{GL}(V)$ whose Lie algebra $\mathscr{G}$ has an element J with $J^2=-1$.

The classification of all semisimple linear groups of twistor type is an open problem but the result of Bryant [8] shows that the isotropy group $G \subset \mathrm{GL}(V)$ of a compact irreducible Riemannian symmetric space L/G is a group of twistor type iff L/G is a Hermitian or quaternionic symmetric space or one of the following spaces:

$$\mathrm{SO}_{p+q}/\mathrm{SO}_p\times\mathrm{SO}_q\,,\qquad \mathrm{Sp}_{p+q}/\mathrm{Sp}_p\times\mathrm{Sp}_q\,,$$
$$\mathrm{E}_7/\mathrm{SO}_8\,,\qquad \mathrm{E}_8/\mathrm{Spin}_{16}\,,\qquad \mathrm{F}_4/\mathrm{Spin}_9\,.$$

These are exactly the symmetric spaces that admit holomorphic twistor spaces.

Let G be a group of twistor type. For any G-structure $\pi:P\to M$ with a connection, we define the twistor space Z as total space of a bundle $\tau:Z\to M$ over M, with a complex affine symmetric space $S=G/H$ as fiber. We define an H-structure with a connection ω on Z, an almost complex structure J_Z that is parallel with respect to the connection ω and a J_Z-invariant ω-parallel distribution $\mathscr{H}_Z$, that is the horizontal distribution of a connection in the bundle τ. Note that our construction of twistor space can be considered as a special form of the construction of Bérard-Bergery and Ochiai.

We show that the conditions of integrability of the almost complex structure J_Z and the holomorphicity of the distribution $\mathscr{H}_Z$ reduce to some linear conditions on the curvature tensor of the connection. They may be considered as generalizations of the self-dual equation. For some Lie group G, these conditions are fulfilled automatically.

In particular, we prove that, if $\pi:P\to M=P/G$ is the G-structure associated with an affine symmetric space P/G of a Lie group P and the isotropy group $G\subset\mathrm{GL}(V)$ is a group of the twistor type, then the associated twistor space Z has an integrable complex structure and the holomorphic distribution $\mathscr{H}_Z$. This can be considered as a generalization of the results of Bryant, since in the case of a Riemannian symmetric manifold P/G, our twistor space coincides with the holomorphic twistor space, constructed by Bryant.

2. Groups of twistor type and symmetric spaces of complex structures

Definition 2.1. A connected linear Lie group $G\subset\mathrm{GL}(V)$, $V=\mathbb{R}^{2n}$, is called a group of twistor type if its Lie algebra $\mathscr{G}$ has an element J with $J^2=-1$. (In other words, J is a complex structure in the vector space V.)

We fix such an element J and we denote by f (by m) the subspace of elements $X\in\mathscr{G}$ that commute (anticommute) with J. Then $m=[J,\mathscr{G}]$ and

$$\mathscr{G}=f+m \tag{2.1}$$

is a symmetric decomposition of the Lie algebra $\mathscr{G}$, that is,

$$[f,m]\subset m\,,\qquad [m,m]\subset f\,,\qquad [f,f]\subset f\,.$$

The left multiplication by J defines, in the space m of endomorphisms, an ad f-invariant complex structure J_m,

$$J_m : X \mapsto JX\,, \quad X \in m\,.$$

The symmetric decomposition (2.1) corresponds to the affine symmetric space

$$S = (\mathrm{Ad}\ G)J = G/H\,, \quad H = Z_G(J)\,,$$

and the operator J_m defines in S an invariant complex structure J_S.

We say that $S = (\mathrm{Ad}\ G)J = G/H$ is the *complex symmetric space associated* with a group G of twistor type and a complex structure $J \in \mathscr{G}$.

Example 2.2. Let $G = \mathrm{GL}_{2n}^+(\mathbb{R}) = \{A \in \mathrm{GL}_{2n}(\mathbb{R}),\ \det A > 0\}$ and let $J \in \mathrm{gl}_{2n}(\mathbb{R})$ be a complex structure in $V = \mathbb{R}^{2n}$. Then the associated complex symmetric space

$$S = \mathrm{Com}_{2n}^+ = \mathrm{GL}_{2n}^+(\mathbb{R})/\mathrm{GL}_n(\mathbb{C})$$

consists of all complex structures J in V, with fixed orientation.

Proposition 2.3. *Let G be a group of twistor type and $S = (\mathrm{Ad}\ G)J$ be the associated symmetric space. Then S is a totally geodesic complex submanifold in* Com_{2n}^+. *Conversely, any totally geodesic complex submanifold $S \subset \mathrm{Com}_{2n}^+$ is a symmetric space associated with some group G of twistor type.*

Note that a complex structure $J \in \mathscr{G}$ can be expressed as

$$J = \exp \tfrac{1}{2}\pi J\,.$$

Hence, it belongs to the Lie group G. More generally

$$S = (\mathrm{Ad}\ G)J \subset G \cap \mathscr{G}\,.$$

This allows us to give the following:

Definition 2.4.

(1) The closed subgroup $G(S)$ of the group G, generated by the set S, is called the group of transvections of the symmetric space $S = (\mathrm{Ad}\ G)J = G/H$.

(2) The closed subgroup $\tilde{G}(S)$ of G, generated by $G(S)$ and the one-parameter subgroup $\exp tJ$, is called the extended group of transvections.

Proposition 2.5. *The groups $G(S)$, $\tilde{G}(S)$ are normal subgroups of G with the Lie algebras*

$$\mathscr{G}(S) = [m, m] + m\,, \qquad \tilde{\mathscr{G}}(S) = \mathbb{R}J + [m, m] + m.$$

The group $\tilde{G}(S)$ is connected and $\pi_0(G(S)) = 1$, $\mathbb{Z}_2$ or $\mathbb{Z}_4$.

3. Conditions for integrability of an almost complex structure and holomorphicity of a distribution

Let V, W be vector spaces with fixed complex structures J. Then the space of W-valued exterior k-forms $C^k(W) = W \otimes \wedge^k V^*$ can be written as

$$C^k(W) = \sum C^{p,q}(W) ,$$

where $C^{p,q}(W)$ is the space of forms of the type (p, q).

Denote by $\Pi^{p,q}: C^k(W) \to C^{p,q}(W)$ the natural projectors. In particular, the projector $\Pi^{02} = N_J$ is given explicitly by the following Nijenhuis formula:

$$N_J(A)(x, y) = \tfrac{1}{4}[A(x, y) + JA(Jx, y) + JA(x, Jy) - A(Jx, Jy)] ,$$
$$A \in C^2(W) , \quad x, y \in V .$$

Consider the following examples. Let $V = W = \mathscr{D}(Z)$ be the space of vector fields on a manifold Z with an almost complex structure J and $A = [\cdot, \cdot]$ be the Lie bracket. Then $N_J([\cdot, \cdot]) = [\![J, J]\!]$ (the Nijenhuis bracket of the vector-valued one-form J).

Suppose, in addition, that on Z there is a linear connection ∇ with torsion tensor T such that $\nabla J = 0$. Then

$$N_J(T) = [\![J, J]\!] .$$

Hence, the Newlander–Nirenberg condition of integrability can be reformulated as follows:

Proposition 3.1. *Let J be an almost complex structure on a manifold Z and ∇ be a connection with torsion tensor T such that $\nabla J = 0$. The structure J is integrable iff $T^{02} \equiv N_J(T) = 0$.*

Let Z be a manifold with an almost complex structure J and let $\mathscr{H}$ be a J-invariant distribution. We assume that there exists a connection ∇ with torsion tensor T that preserves J and $\mathscr{H}$ ($\nabla J = 0$, $\nabla \mathscr{H} \subset \mathscr{H}$).

The operator J defines a complex structure in the space $V = \mathscr{D}(Z)$ of vector fields. The vector fields that belong to the distribution $\mathscr{H}$ form a complex subspace $U = \mathscr{D}(\mathscr{H})$. Define the two-form

$$S: V \wedge V \to V/U$$

by the formula

$$S(X, Y) = T(X, Y) \mod U .$$

It can be composed into a sum of pure components,

$$S = S^{20} + S^{11} + S^{02} .$$

Proposition 3.2. *The following conditions are equivalent:*

(1) $S^{11}(JX, \cdot) = JS^{11}(X, \cdot)$ *for any* $X \in U$.

(2) The distribution $\mathscr{K}$ *can be defined locally by the equations* $\eta_1 = \cdots = \eta_m = 0$, *where* η_i *are forms of the type* $(1,0)$, *and the* $(1,1)$ *component* $(\mathrm{d}\eta_i)^{11}$ *of their differential belongs to the ideal* $(\eta_1, \ldots, \eta_m)$.

Definition 3.3. We say that a J-invariant distribution $\mathscr{K}$ is almost holomorphic if the equivalent conditions of proposition 3.2 are fulfilled.

Theorem 3.4. *Suppose that an almost complex structure on a manifold Z is integrable. Then any almost holomorphic distribution on Z is holomorphic, i.e., it can be defined by a set of holomorphic one-forms.*

4. G-structures with a connection and derived H-structures

Let $G \subset \mathrm{GL}(V)$, $V = \mathbb{R}^n$, be a linear group.

Definition 4.1. A principal G-bundle $\pi: P \to M$ over an n-dimensional manifold M (with the left action of G) is called a *G-structure* if it is equipped with a V-valued G-equivariant strictly horizontal one-form $\theta: TP \to V$ (a one-form on P is called strictly horizontal if its kernel is the vertical subbundle of the tangent bundle TP). The form θ is called the *displacement form* of a G-structure.

This definition is equivalent to the standard one. Indeed, the one-form θ_p in a point $p \in P$ defines, in an obvious way, an isomorphism $\hat{p}: T_{\pi p}M \to V$ that is a coframe of M. Se we may identify P with a principal G-subbundle $\hat{P} = \{\hat{p}, p \in P\}$ of the coframe bundle.

Let $\pi: P \to M$ be a G-structure with displacement form θ. For any closed subgroup H of G we define a principal H-bundle

$$\pi': P \to Z = P/H,$$

where $Z = P/H$ is the orbit space of the group H into P, and a bundle

$$\tau: Z = P/H \to M = P/G$$

with a fiber G/H. The bundle τ is the bundle associated with the principal bundle π and the natural action of G on G/H:

$$\tau: Z = P \times_G G/H \to P/G = M.$$

Choose a connection $\omega: TP \to \mathscr{G} = \mathrm{Lie}\ G$ in the G-structure π and assume that the subgroup H is reductive, i.e., that there exists a reductive decomposition

$$\mathscr{G} = f + m, \quad [f, m] \subset m, \tag{4.1}$$

of the Lie algebra $\mathscr{G}$, where f is the Lie algebra of H.

Then we can turn π' into an H-structure with a connection. To do this, first of all, we identify H with a subgroup of the linear group $\mathrm{gl}(V')$ of the vector space $V' = V \oplus m$ by associating with an element $h \in H$ the transformation

$$h: V' \ni (v, X) \mapsto (hv, [h, X]) .$$

For any $\mathscr{G}$-valued form α on P we shall denote by α^f and α^m its f and m components, respectively,

$$\alpha = \alpha^f + \alpha^m .$$

The following result can be checked immediately.

Theorem 4.2. *Let $\pi: P \to M$ be a G-structure with a connection and let $\theta: TP \to V$ and $\omega: TP \to \mathscr{G}$ be the displacement form and the connection form, respectively. Let H be a reductive subgroup of G with a reductive decomposition (4.1). Then the principal H-bundle*

$$\pi': P \to P/H = Z$$

equipped with the $V' = V \oplus m$-valued one-form $\theta' = \theta + \omega^m$, is an H-structure and the one-form $\omega' = \omega^f$ is a connection form on this H-structure.

Definition 4.3. The H-structure $\pi': P \to P/H = Z$ with the connection ω' is called the derived H-structure of a G-structure π. The projection $\mathscr{H}_Z = \pi'_*(\mathrm{Ker}\ \omega)$ on Z of the horizontal distribution $\mathscr{H} = \mathrm{Ker}\ \omega$ of the connection ω is called the canonical (horizontal) distribution on Z.

Note that $\mathscr{H}_Z$ is the horizontal distribution of the connection in the associated bundle $\tau: Z \to M$ induced by the connection ω. The curvature and the torsion forms Ω', Θ' of the derived connection ω' can be expressed in terms of the curvature and the torsion forms Ω, Θ of the initial connection ω as follows:

$$\Theta' = \begin{pmatrix} \Theta - \omega^m \wedge \theta \\ \Omega^m - \frac{1}{2}[\omega^m \wedge \omega^m]^m \end{pmatrix}, \qquad \Omega' = \begin{pmatrix} \Psi & 0 \\ 0 & \mathrm{ad}\ \Psi \end{pmatrix}, \tag{4.2}$$

where the vectors from $V' = V \oplus m$ are considered as columns $v' = \binom{v}{X}$, $v \in V$, $X \in m$, and $\psi = \Omega^f - \frac{1}{2}(\mathrm{ad}\ \omega^m \wedge \omega^m)^f$.

5. The twistor space of a G-structure of twistor type

Now let $\pi: P \to M$ be a G-structure with a connection where $G \subset \mathrm{GL}(V)$, $V = \mathbb{R}^{2n}$ is a Lie group of twistor type. We call this G-structure with a connection a G-

structure of twistor type. We fix a complex structure $J_0 \in \mathcal{G}$ in V and denote by $H = Z_G(J_0)$ its centralizer in G. As in section 4, we identify H with a subgroup of the group $\mathrm{GL}(V')$, $V' = V \oplus m$.

Note that the group H commutes with the complex structure $J' = J_0 \oplus J_m$ of the space V'. Hence, $H \subset \mathrm{GL}_{n'}(\mathbb{C})$, $n' = \frac{1}{2}\dim V'$. This means that the H-structure $\pi' : P \to Z = P/H$ is subordinated to the $\mathrm{GL}_{n'}(\mathbb{C})$-structure that is an almost complex structure J_Z on Z. This almost complex structure takes the standard form with respect to any coframe $\hat{p} : T_z Z \to V'$ associated with an element $p \in P$. More precisely,

$$(J_Z)_z = \hat{p}^{-1} \circ J' \circ \hat{p}\,, \quad z = \pi' p \in Z\,.$$

Definition 5.1. Let $\pi : P \to M$ be a G-structure of twistor type and $H = Z_G(J_0)$ the centralizer of a complex structure $J_0 \in \mathcal{G}$ into G. The manifold $Z = P/H$ equipped with the canonical distribution $\mathcal{H}_Z$, the derived H-structure $\pi' : P \to Z$ with connection $\omega' = \omega^f$ and the almost complex structure J_Z, is called the *twistor space* of the G-structure π associated with a complex structure $J_0 \in \mathcal{G}$.

We remark that the distribution $\mathcal{H}_Z$ and the almost complex structure J_Z are parallel with respect to the connection ω'. Hence, we can apply the result of section 3. Using (4.2), we obtain:

Theorem 5.2. *Let $\pi : P \to M$ be a G-structure of the twistor type with connection form ω. Then*

[i] the almost complex structure J_Z on the twistor space $Z = P/H$ is integrable iff

$$\Theta^{02} = 0\,, \qquad (\Omega^m)^{02} = 0\,; \tag{i}$$

[h] the canonical distribution $\mathcal{H}_Z$ is almost holomorphic iff

$$(\Omega^m)^{11} = 0\,, \tag{h}$$

where Θ, Ω are the torsion form and the curvature form of the connection ω, $\Omega = \Omega^f + \Omega^m$ is the decomposition of Ω associated with the symmetric decomposition

$$\mathcal{G} = f + m\,, \quad \mathcal{G} = \mathrm{Lie}\, G\,, \quad f = \mathrm{Lie}\, H\,,$$

and, for any horizontal form α with values in the complex space (V', J'), we denote by α^{pq} its (p, q) component.

6. The space $\mathfrak{R}(\mathcal{G})$ of curvature tensors

In this basic purely algebraic section we study the space $\mathfrak{R}(\mathcal{G})$ of algebraic curvature tensors of type $\mathcal{G}$, where $\mathcal{G}$ is the Lie algebra of a group G of twistor type.

In other words, $\mathfrak{R}(\mathscr{G})$ is the space of $\mathscr{G}$-valued exterior two-forms, closed under the Koszul–Spencer differential.

We describe the decomposition of the space $\mathfrak{R}(\mathscr{G})$, associated with a complex structure $J\in\mathscr{G}$, and define two important spaces,

$$\mathfrak{R}_+(m)=\Pi_{01}^{02}\mathfrak{R}(\mathscr{G})=\Pi^{02}\mathfrak{R}(m)\ ,$$

$$\Pi_{01}^{11}\mathfrak{R}(\mathscr{G})\approx\mathfrak{R}_-(\mathscr{G})/\mathfrak{R}_-(f)\ .$$

The obstacles to the integrability of the almost complex structure J_Z on the twistor space Z and the almost holomorphicity of the distribution $\mathscr{H}_Z$ lie in these spaces.

We also define two G-submodules $\mathfrak{R}_{\text{int}}(\mathscr{G})$ and $\mathfrak{R}_{\text{hol}}(\mathscr{G})$. The complex structure J_Z is integrable iff the curvature tensor of the corresponding connection ω belongs to $\mathfrak{R}_{\text{int}}(\mathscr{G})$ and the distribution $\mathscr{H}_Z$ is almost holomorphic iff it belongs to $\mathfrak{R}_{\text{hol}}(\mathscr{G})$. We prove some results about these spaces which imply the conditions of integrability of J_Z and of the holomorphicity of $\mathscr{H}_Z$ obtained in the next section.

Let $V=\mathbb{R}^{2n}$ be a vector space with a complex structure J_0. Also we denote by J_0 the induced complex structure in $\mathrm{gl}(V)$,

$$J_0: X\mapsto J_0X\ ,\quad X\in\mathrm{gl}(V)\ ,$$

and by $\mathrm{gl}_{1,0}$ $[\mathrm{gl}_{0,1}]$ the J_0-invariant subspace of elements $X\in\mathrm{gl}(V)$ that commute [anticommute] with J_0. In other words, $\mathrm{gl}_{1,0}$ $[\mathrm{gl}_{0,1}]$ is the space of V-valued one-forms of type $(1,0)$ $[(0,1)]$.

Let $\mathfrak{G}\subset\mathrm{gl}(V)$ be a J_0-invariant subspace. We have the following decomposition of the space $C^2(\mathfrak{G})$ of $\mathfrak{G}$-valued two-forms:

$$C^2(\mathfrak{G})=\sum_{p+q=2}C^{pq}(\mathfrak{G})\ , \tag{6.1}$$

where $C^{pq}(\mathfrak{G})$ is the space of forms of type (p,q). In particular, we have

$$C^2=C^2(\mathrm{gl}(V))=\sum_{p+q=2}C_{10}^{pq}+\sum_{p+q=2}C_{01}^{pq}\ ,$$

$$C_{10}^{pq}=C^{pq}(\mathrm{gl}_{1,0})\ ,\qquad C_{01}^{pq}=C^{pq}(\mathrm{gl}_{0,1})\ .$$

Denote by

$$\Pi_{rs}: C^2\to C^2(\mathrm{gl}_{r,s})\ ,\qquad \Pi^{pq}: C^2\to C^{pq}\ ,\qquad \Pi_{rs}^{pq}: C^2\to C_{rs}^{pq}$$

the natural projections and by

$$d: C^2=V\otimes V^*\otimes\wedge^2V^*\to V\otimes\wedge^3V^*$$

the natural map defined by alternation. We remark that d is a particular case of the Koszul–Spencer differential,

$$d: V\otimes S^kV^*\otimes\wedge^lV^*\to V\otimes S^{k-1}V^*\otimes\wedge^{l+1}V^*\ ,$$

of the complex

$$\left(\sum V\otimes S^k V^*\otimes S^l V^*, d\right).$$

Definition 6.1. Let $\mathfrak{G}\subset \mathrm{gl}(V)$ be a subspace. The space

$$\mathfrak{R}(\mathfrak{G})=\{R\in C^2(\mathfrak{G}),\ dR=0\}$$

of d-closed $\mathfrak{G}$-valued two-forms is called the space of curvature tensors of type $\mathfrak{G}$. For any $A\in \mathrm{gl}(V)$ $[B\in \mathrm{GL}(V)]$ we denote by $\hat{A}$ $[T_B]$ the associated derivation [automorphism] of the tensor algebra $\mathrm{Ten}(V)$.

Let $\mathfrak{G}\subset \mathrm{gl}(V)$ be an ad J_0-invariant subspace. Then

$$\mathfrak{G}=f\oplus m\,, \tag{6.2}$$

where $f=Z_{\mathfrak{G}}(J_0)\subset \mathrm{gl}_{1,0}$ is the centralizer and $m=[J_0,\mathfrak{G}]\subset \mathrm{gl}_{0,1}$. We have the following obvious lemma:

Lemma 6.2.

(1) The operators $\hat{J}_0$, $T_{J_0}=\exp\frac{1}{2}\pi\hat{J}_0$, $T_{\sqrt{J_0}}$, *where* $\sqrt{J_0}=(1/\sqrt{2})(1+J_0)$, *preserve the spaces* $\mathfrak{R}(\mathfrak{G})$, $\mathfrak{R}(f)$, $\mathfrak{R}(m)$.

(2) $\mathfrak{R}(\mathfrak{G})=\mathfrak{R}_+(\mathfrak{G})+\mathfrak{R}_-(\mathfrak{G})$,
where

$$\mathfrak{R}_\pm(\mathfrak{G})=\{R_\pm\equiv\Pi_\pm R:=\tfrac{1}{2}[R(\cdot,\cdot)\mp J_0R(J_0\cdot,J_0\cdot)J_0]\,,\ R\in\mathfrak{R}(\mathfrak{G})\}$$

is the eigenspace of the operator T_{J_0} *with eigenvalue* ± 1.

For any J_0-invariant subspace $\mathfrak{G}\subset \mathrm{gl}(V)$, we set

$$\mathfrak{R}_{\mathbb{C}}(\mathfrak{G})=\{R\in\mathfrak{R}(\mathfrak{G}),\ R(J_0\cdot,\cdot)=R(\cdot,J_0\cdot)=J_0R(\cdot,\cdot)=R(\cdot,\cdot)J_0\}\,.$$

We say that $\mathfrak{R}_{\mathbb{C}}(\mathfrak{G})$ is the space of holomorphic curvature tensors of type $\mathfrak{G}$.

We have the following important proposition:

Proposition 6.3. *Let* $\mathfrak{G}\subset \mathrm{gl}(V)$ *be an* ad J_0*-invariant subspace with the decomposition* (6.2). *Then*

(1)
$$\begin{aligned}\mathfrak{R}_+(\mathfrak{G})&=(\Pi^{02}_{01}+\Pi^{20}_{01}+\Pi^{11}_{10})\mathfrak{R}(\mathfrak{G})\\&=\Pi^{02}_{01}\mathfrak{R}(\mathfrak{G})\oplus(\Pi^{20}_{01}+\Pi^{11}_{10})\mathfrak{R}(\mathfrak{G})\\&=\Pi^{02}_{01}\mathfrak{R}(\mathfrak{G})\oplus\mathrm{Ker}(\hat{J}|_{\mathfrak{R}(\mathfrak{G})})\,;\end{aligned}$$

(2)
$$\Pi^{02}_{01}\mathfrak{R}(\mathfrak{G})=\Pi^{02}\mathfrak{R}(m)=\mathfrak{R}_+(m)$$

where $\mathfrak{R}_+(m)=\{R\in\mathfrak{R}(m),\ T_{J_0}R=R\}$;

$$(3) \qquad \mathfrak{R}_-(\mathfrak{G}) \subset \mathfrak{R}(\mathfrak{G}+J_0\mathfrak{G}) = \Pi^{20}_{10}\mathfrak{R}(\mathfrak{G}) \oplus (\Pi^{02}_{10}+\Pi^{11}_{01})\mathfrak{R}(\mathfrak{G}) ,$$

$$\Pi^{20}_{10}\mathfrak{R}(\mathfrak{G}) = \Pi^{20}_{10}\mathfrak{R}(J_0\mathfrak{G}) ,$$

$$(\Pi^{02}_{10}+\Pi^{11}_{01})\mathfrak{R}(\mathfrak{G}) = (\Pi^{02}_{10}+\Pi^{11}_{01})\mathfrak{R}(J_0\mathfrak{G}) \approx \Pi^{11}_{01}\mathfrak{R}(\mathfrak{G}) ;$$

$$(4) \qquad \Pi^{11}_{01}\mathfrak{R}(\mathfrak{G}) \simeq \mathfrak{R}_-(\mathfrak{G})/\mathfrak{R}_-(f) ,$$

where

$$\mathfrak{R}_-(f) = \{R \in \mathfrak{R}(f) , T_{J_0}R = -R\} ,$$

$$\mathfrak{R}_-(f) = \Pi^{20}_{10}\mathfrak{R}(\mathfrak{G}) \cap \mathfrak{R}(\mathfrak{G}) = \mathfrak{R}_{\mathbb{C}}(\mathfrak{G} \cap J_0\mathfrak{G}) .$$

The proof follows from the study of the eigenvalues of the operators $\hat{J}_0$, T_{J_0}, $T_{\sqrt{J_0}}$ and $J_0+2\hat{J}_0$ in the space C^{pq}_{rs}.

We have the following canonical decomposition of a tensor $R \in \mathfrak{R}(\mathfrak{G})$:

$$R = [Q^{02} + (Q^{20}+P^{11})] + [P^{20} + (P^{02}+Q^{11})] = R_+ + R_- ,$$

where $P^{r,s} \in C^{rs}_{10}$, $Q^{rs} \in C^{rs}_{01}$, $R_\pm \in \mathfrak{R}_\pm(\mathfrak{G})$. Using the Nijenhuis formula from section 3, we can derive the following expression for Q^{02}:

$$\begin{aligned} Q^{02} &= \Pi^{02}_{01}R = \Pi^{02}R_+ \\ &= \tfrac{1}{8}[R(\cdot,\cdot) + JR(J\cdot,\cdot) + JR(\cdot,J\cdot) - R(J\cdot,J\cdot) \\ &\quad - JR(J\cdot,J\cdot)J - R(\cdot,J\cdot)J - R(J\cdot,\cdot)J + JR(\cdot,\cdot)J] . \end{aligned}$$

Proposition 6.3 implies the following:

Corollary 6.4. *Let* $\mathfrak{G} = f + m \subset \mathrm{gl}(V)$ *be an* ad J_0*-invariant subspace. Then for any* $R \in \mathfrak{R}(\mathfrak{G})$ *we have*

$$Q^{02} \in \mathfrak{R}(m) , \qquad Q^{20}+P^{11} \in \mathfrak{R}(\mathfrak{G}) ,$$

$$P^{20} \in \mathfrak{R}(f+J_0f) , \qquad P^{02}+Q^{11} \in \mathfrak{R}(\mathfrak{G}+J_0\mathfrak{G}) ,$$

$$R_\pm \in \mathfrak{R}(\mathfrak{G}) .$$

Corollary 6.5. *Assume that*

$$\mathfrak{R}_{\mathbb{C}}(f \cap J_0 f) \equiv \mathfrak{R}_-(f) = 0 .$$

This is the case if $f \cap J_0 f = 0$ *or* $n = \frac{1}{2}\dim V \geq 3$ *and* $f \cap J_0 f = \mathbb{R}\,\mathrm{id} + \mathbb{R}J_0$. *Then*

$$\Pi^{11}_{01}\mathfrak{R}(\mathfrak{G}) \simeq \mathfrak{R}_-(\mathfrak{G}) .$$

Let $\mathscr{G}$ be the Lie algebra of a group G of twistor type and $J_0 \in \mathscr{G}$ a complex structure. Then the space $\mathscr{G}$ is ad J-invariant for any complex structure $J \in S = (\mathrm{Ad}\ G)J_0$. Hence, we can apply the above constructions and results to the case when V is equipped with an arbitrary complex structure $J \in S$. In this case we shall add the letter J to the appropriate symbols. For example, the decomposition (6.1) with respect to a complex structure J is written as

$$C^2(\mathfrak{G}) = \sum_{p+q=2} C^{pq}_{(J)}(\mathfrak{G}) .$$

Now we define two subspaces of $\mathfrak{R}(\mathscr{G})$:

$$\mathfrak{R}_{\mathrm{int}}(\mathscr{G}) = \bigcap_{J \in S} \mathrm{Ker}\, \Pi^{02}_{(J)01} ,$$

$$\mathfrak{R}_{\mathrm{hol}}(\mathscr{G}) = \bigcap_{J \in S} \mathrm{Ker}\, \Pi^{11}_{(J)01} .$$

Lemma 6.6. *The spaces $\mathfrak{R}_{\mathrm{int}}(\mathscr{G})$ and $\mathfrak{R}_{\mathrm{hol}}(\mathscr{G})$ are invariant under the action of the group G into the space $\mathfrak{R}(\mathscr{G})$.*

Now we give some characterization of these spaces. First of all, proposition 6.3 shows that

$$\mathfrak{R}_{\mathrm{int}}(\mathscr{G}) = \{R \in \mathfrak{R}(\mathscr{G}),\ \hat{J} \cdot R^{(J)}_{+} = 0,\ \forall J \in S\} ,$$

$$\mathfrak{R}_{\mathrm{hol}}(\mathscr{G}) = \{R \in \mathfrak{R}(\mathscr{G}),\ R^{(J)}_{-} \in \mathfrak{R}^{(J)}_{-}(f_J),\ \forall J \in S\} ,$$

where $R = R^{(J)}_{+} + R^{(J)}_{-}$ is the decomposition of R into the sum of eigenvectors of the operator T_J with eigenvalues ± 1 and

$$\mathfrak{R}^{(J)}_{-}(f_J) = \{R \in \mathfrak{R}(f_J),\ T_J R = -R\} , \quad f_J = Z_{\mathscr{G}}(J) .$$

For any group K that acts linearly on a vector space W, we denote by W^K the space of K-invariant vectors.

Proposition 6.7. *Let G be a group of twistor type, $S = (\mathrm{Ad}\ G)J_0$ the symmetric space associated with a complex structure $J_0 \in \mathscr{G}$ and $G(S)$ and $\tilde{G}(S)$ the group of transvections and the extended group of transvections. Then*

(1) $\mathfrak{R}(\mathscr{G})^{G(S)} \subset \mathfrak{R}_{\mathrm{hol}}(\mathscr{G})$;

(2) $\mathfrak{R}(\mathscr{G})^{G(S)} \cap \mathfrak{R}_{\mathrm{int}}(\mathscr{G}) = \mathfrak{R}(\mathscr{G})^{\tilde{G}(S)}$;

(3) assume that $\mathfrak{R}_{\mathbb{C}}(f \cap J_0 f) \equiv \mathfrak{R}_{-}(f) = 0$, where $f = Z_{\mathscr{G}}(J_0)$.

This is the case if $f \cap J_0 f = 0$ or $f \cap J_0 f = \mathbb{R}\ \mathrm{id} + \mathbb{R} J_0$ and $n = \frac{1}{2}\dim V \geqslant 3$. Then

$$\mathfrak{R}_{\mathrm{hol}}(\mathscr{G}) = \mathfrak{R}(\mathscr{G})^{G(S)} .$$

Proof.

(1) Let $R \in \mathfrak{R}(\mathscr{G})^{G(S)}$. Then for any $J \in S \subset G(S)$ we have $T_J R = R$, that is,

$$R=R_+\ ,\qquad 0=R_-=P^{20}_{(J)}+P^{02}_{(J)}+Q^{11}_{(J)}\ .$$

Hence, $Q^{11}_{(J)}=\Pi^{11}_{(J)01}R=0\ \forall J$ and $R\in\mathfrak{R}_{\mathrm{hol}}(\mathscr{G})$.

(2) By proposition 6.3(1), the following conditions are equivalent:

(a) $R\in\operatorname{Ker}\Pi^{02}_{(J)01}$,

(b) $\hat{J}R=0$,

(c) $\{\exp t\hat{J}\}R=\{T_{\exp TJ}\}R=R$.

But $\tilde{G}(S)=\{\exp tJ\}\cdot G(S)$ and we may assume that the tensor R is $G(S)$-invariant. This proves (2).

(3) Assume that $\mathfrak{R}_-(f)=0$, where $f=Z_{\mathscr{G}}(J_0)$. Then a similar equality is true for any complex structure $J\in S$ and the associated decomposition $\mathscr{G}=f_{(J)}+m_{(J)}$. By proposition 6.3(4), we have in this case

$$\Pi^{11}_{(J)01}\mathfrak{R}(\mathscr{G})\simeq\mathfrak{R}^{(J)}_-(\mathscr{G})\ .$$

If $R\in\mathfrak{R}_{\mathrm{hol}}(\mathscr{G})=\bigcap_J\operatorname{Ker}\Pi^{11}_{(J)01}$, then $R^{(J)}_-=0$, $R=R^{(J)}_+$. This means that $T_JR=R$ for any $J\in S$. Since $G(S)$ is generated by $J\in S$, the tensor R is $G(S)$-invariant. □

Proposition 6.8. *The following conditions are equivalent:*

(a) $\mathfrak{R}_{\mathrm{int}}(\mathscr{G})=\mathfrak{R}(\mathscr{G})$,

(b) $\mathfrak{R}(m)\subset\operatorname{Ker}\Pi^{02}$,

(c) $\mathfrak{R}(m)\cap\operatorname{Im}\Pi^{02}=\{0\}$,

where Π^{02} *is the projector associated with the complex structure* J_0 *and* $m=[J_0,\mathscr{G}]$.

The proof follows from the definitions.

Now we give some conditions for $\mathfrak{R}_{\mathrm{int}}(\mathscr{G})\neq 0$. We denote by

$$\mathfrak{R}_0(\mathscr{G})=\{R\in R(\mathscr{G}),\ \mathrm{ric}(R)=0\}$$

the subspace of traceless curvature tensors, where $\mathrm{ric}(R)\in\otimes^2V^*$ is the Ricci tensor of R defined by

$$\mathrm{ric}(R)(x,y)=\mathrm{tr}(z\mapsto R(x,z)y)\ .$$

Note that the *k-prolongation* of a subspace $\mathfrak{G}\subset\mathrm{gl}(V)$ is defined as

$$\mathfrak{G}^{(k)}=(\mathfrak{G}\otimes S^{k-1}V^*)\cap(V\otimes S^kV^*)\ .$$

Proposition 6.9.

(1) Let $\mathscr{D}\subset\mathfrak{R}(\mathscr{G})$ *be a* $\tilde{G}(S)$*-invariant subspace with* $\mathscr{D}\cap\mathfrak{R}^0(\mathscr{G})=\{0\}$. *Then* $\mathscr{D}\subset\mathfrak{R}_{\mathrm{int}}(\mathscr{G})$.

(2) Assume that $\mathscr{G}^{(1)}\neq\{0\}$ *and* $m^{(1)}=\{0\}$, *where* $m=[J_0,\mathscr{G}]$. *Then*

$$\mathfrak{R}_{\mathrm{int}}(\mathscr{G})\supset d(\mathscr{G}^{(1)}\otimes V^*)\neq\{0\}\ .$$

(3) Let $\mathscr{G}$ be a reductive Lie algebra of type two (that is, $\mathscr{G}^{(1)} \neq 0$, $\mathscr{G}^{(2)} = 0$). Then

$$\mathfrak{R}_{\text{int}}(\mathscr{G}) \supset d(\mathscr{G}^{(1)} \otimes V^*) \simeq \mathscr{G}^{(1)} \otimes V^*.$$

Now we assume that there is a G-invariant complement to the subspace $\mathfrak{R}_0(\mathscr{G})$ in $\mathfrak{R}(\mathscr{G})$. (This is always the case if the Lie algebra $\mathscr{G}$ is semisimple.) We denote this complement by $\mathfrak{R}_{\text{ric}}$ since the map

$$\text{ric}: \mathfrak{R}_{\text{ric}} \to \text{ric}(\mathfrak{R}(\mathscr{G}))$$

is an isomorphism and tensors from $\mathfrak{R}_{\text{ric}}$ are determined by their Ricci tensors.

Definition 6.10. The projection $W(R)$ of a tensor $R \in \mathfrak{R}(\mathscr{G}) = \mathfrak{R}_0(\mathscr{G}) \oplus \mathfrak{R}_{\text{ric}}$ on the subspace $\mathfrak{R}_0(\mathscr{G})$ is called the Weyl tensor of R.

Proposition 6.11. *Let $\mathscr{G}$ be the Lie algebra of a group G of twistor type. Assume that there is G-invariant decomposition*

$$\mathfrak{R}(\mathscr{G}) = \mathfrak{R}_0(\mathscr{G}) \oplus \mathfrak{R}_{\text{ric}}.$$

(1) Then $\mathfrak{R}_{\text{ric}}(\mathscr{G}) \subset \mathfrak{R}_{\text{int}}(\mathscr{G})$.

(2) Assume that the G-module $\mathfrak{R}_0(\mathscr{G})$ is irreducible and that $\mathfrak{R}_+(m) \neq 0$. Then $\mathfrak{R}_{\text{ric}}(\mathscr{G}) = \mathfrak{R}_{\text{int}}(\mathscr{G})$.

(3) Assume that $\mathfrak{R}_0(\mathscr{G}) = \mathscr{W}_+ + \mathscr{W}_-$, where the G-module $\mathscr{W}_-$ consists of $\tilde{G}(S)$-invariant tensors and the G-module $\mathscr{W}_+$ is irreducible, and $\mathfrak{R}_+(m) \neq 0$.
Then

$$\mathfrak{R}_{\text{int}}(\mathscr{G}) = \mathfrak{R}_{\text{ric}} \oplus \mathscr{W}_-.$$

Proof. (1) and (2) follow from proposition 6.9(1), and (3) follows from (1) and proposition 6.7(2). □

7. Conditions of integrability of the almost complex structure and of holomorphicity of the distribution in the twistor space

First of all we reformulate the main theorem 5.2 in more convenient terms. Let $\pi: P \to M$ be a G-structure with the displacement form θ and let Ω be the curvature form of a connection ω in π. We can associate with Ω a function R^{ω} with values in the space $C^2(\mathscr{G})$ of $\mathscr{G}$-valued two-forms as

$$R^{\omega}: P \ni p \mapsto R^{\omega}_p = \Omega_p(\hat{p}^{-1}\cdot, \hat{p}^{-1}\cdot),$$

where $\hat{p}: T_{\pi p}M \to V$ is a coframe, defined by $p \in P$. R^{ω} is called the curvature function of a connection ω. Note that its value R^{ω}_p is the curvature tensor of the connection ω calculated with respect to the coframe $\hat{p}$. For simplicity, we shall assume from now on that *the connection ω has no torsion.* In this case the curvature function R takes its values in the space $\mathfrak{R}(\mathscr{G})$.

We can reformulate theorem 5.2 as follows:

Theorem 7.1. *Let $\pi:P\to M$ be a G-structure of twistor type with a torsionless connection ω and let $R^{\omega}:P\to\mathfrak{R}(\mathscr{G})$ be its curvature function. Let Z be the twistor space of π associated with a complex structure $J_0\in\mathscr{G}$. Then*

[i] the almost complex structure J_Z on the associated twistor space Z is integrable if and only if

$$Q^{02}_p := \Pi^{02}_{01} R^{\omega}_p = 0 \quad \forall p \in P, \tag{i}$$

[h] the canonical distribution $\mathscr{H}_Z$ on Z is almost holomorphic if and only if

$$Q^{11}_p := \Pi^{11}_{01} R^{\omega}_p = 0 \quad \forall p \in P. \tag{h}$$

(Here $\Pi^{pq}_{rs} = \Pi^{pq}_{(J_0)rs}$.)

Let Π_J be one of the projectors $\Pi^{pq}_{(J)rs}$ associated with a complex structure $J\in S=(\mathrm{Ad}\ G)J_0$. Then for any $g\in G$, we have

$$\Pi_{T_gJ} T_g R^{\omega}_p = \Pi_{T_gJ} R^{\omega}_{gp} = T_g(\Pi_J R^{\omega}_p), \tag{7.1}$$

or

$$\Pi_J R^{\omega}_{gp} = \Pi_J T_g R^{\omega}_p = T_g \Pi_{T_g^{-1}J} R^{\omega}_p.$$

This shows that conditions (i), (h) can be rewritten as follows:

$$R^{\omega}_p \in \mathfrak{R}_{\mathrm{int}}(\mathscr{G}) = \bigcap_{J\in S} \mathrm{Ker}\, \Pi^{02}_{(J)01}, \quad \forall p \in P, \tag{i'}$$

$$R^{\omega}_p \in \mathfrak{R}_{\mathrm{hol}}(\mathscr{G}) = \bigcap_{J\in S} \mathrm{Ker}\, \Pi^{11}_{(J)01}, \quad \forall p \in P. \tag{h'}$$

We obtain

Corollary 7.2. *Under the assumptions of theorem* 7.1, *condition [i] is fulfilled iff the curvature function R takes its values in the G-module $\mathfrak{R}_{\mathrm{int}}(\mathscr{G})$, and condition [h] is fulfilled iff R^{ω} takes its values in the G-module $\mathfrak{R}_{\mathrm{hol}}(\mathscr{G})$.*

Now the propositions of section 6 can be reformulated as different conditions for the integrability of the almost complex structure J_Z on the twistor space Z and the almost holomorphicity of the canonical distribution $\mathscr{H}_Z$. We state some of them. Proposition 6.7(1), (2) implies

Theorem 7.3. *Under the assumptions of theorem 7.1, assume that the group of transvections $G(S)$ preserves the curvature tensor:*

$$T_g R_p^{\omega} = R_p^{\omega}, \quad \forall p \in P, \quad \forall g \in G(S).$$

Then

(1) the distribution $\mathcal{H}_Z$ is almost holomorphic,

(2) the almost complex structure J_Z is integrable iff the curvature tensor is invariant under the extended group of transvections $\tilde{G}(S)$.

Corollary 7.4. *Let $M = L/G$ be an affine symmetric space and $\pi: L \to L/G$ be the associated G-structure with connection, where G is considered as a linear group in the isotropy representation. Assume that G is a group of twistor type and Z is the twistor space of π associated with a complex structure $J_0 \in \mathcal{G}$.*

Then, the almost complex structure J_Z on the twistor space Z is integrable and the canonical distribution $\mathcal{H}_Z$ is holomorphic.

Assertion (3) of proposition 6.7 implies the following theorem, which may be considered as a partial reversion of theorem 7.3.

Theorem 7.5. *Under the assumptions of theorem 7.1, assume that*

$$\mathfrak{R}_-(f) \equiv \mathfrak{R}_{\mathbb{C}}(f \cap J_0 f) = 0,$$

where $f = Z_{\mathcal{G}}(J_0)$. This is the case if $f \cap J_0 f = 0$ or $f \cap J_0 f = \mathbb{R}\,\mathrm{id} + \mathbb{R} J_0$ and $n = \frac{1}{2}\dim V \geqslant 3$.

(1) Then the canonical distribution $\mathcal{H}_Z$ in the twistor space Z is almost holomorphic iff the curvature tensor R_p^{ω}, $p \in P$, is invariant under the group of transvections $G(S)$.

(2) If this is the case, the Ricci tensor $\mathrm{ric}(R_{\mathrm{p}}^{\omega})$ *is invariant under the extended group of transvections $\tilde{G}(S)$.*

(3) If the curvature tensor is invariant under the group $\tilde{G}(S)$, then the almost complex structure J_Z is integrable.

Corollary 7.6. *Under the assumptions of theorems 7.1 and 7.5, assume that $G = G(S)$. (This is the case if the group G is simple.) Then the following conditions are equivalent.*

(1) $\mathcal{H}_Z$ is an almost holomorphic distribution.

(2) $\mathcal{H}_Z$ is a holomorphic distribution in the complex twistor space (Z, J_Z).

(3) The curvature tensor in each point is invariant under the group G.

From proposition 6.11 we have

Theorem 7.7. *Under the assumptions of theorem 7.1, assume that*

$$\mathfrak{R}(\mathcal{G}) = \mathfrak{R}_0(\mathcal{G}) \oplus \mathfrak{R}_{\mathrm{ric}},$$

where $\mathfrak{R}_{\rm ric}$ is a G-module. Then the vanishing of the Weyl tensor $W(R^{\omega})$ is sufficient for the integrability of the complex structure J_Z. If the G-module $\mathfrak{R}_0(\mathscr{G})$ is irreducible, the condition $W(R^{\omega})=0$ is also necessary for the integrability of J_Z.

Proposition 6.8 can be reformulated as

Theorem 7.8. *The almost complex structure J_Z on the twistor space Z of an arbitrary G-structure with a torsionless connection is integrable iff the following two equivalent conditions are fulfilled:*

(b) $\mathfrak{R}(m) \subset \operatorname{Ker} \Pi^{02}$

(c) $\mathfrak{R}(m) \cap \operatorname{Im} \Pi^{02}=0$.

This is the case if $\mathfrak{R}(m)=0$.

8. Examples

We apply the general results of sections 6 and 7 to G-structures with some classical irreducible linear Lie groups G of twistor type that belong to the Berger list of irreducible holonomy groups of torsionless connections.

For each group G we indicate the symmetric space $S=G/H$, $H=Z_G(J_0)$, associated with a complex structure $J_0 \in \mathscr{G}$ and the groups $G(S)$, $\tilde{G}(S)$ and we state the conditions that are equivalent to the following conditions:

(i) The almost complex structure J_Z on the twistor space Z, associated with a G-structure π and with torsionless connection ω, is integrable.

(h) The canonical distribution $\mathscr{H}_Z$ on Z is almost holomorphic.

We denote by R^{ω}, ric^{ω}, W^{ω} the curvature tensor, Ricci tensor and Weyl tensor of the connection ω. In some cases, we also describe the decomposition of the G-module $\mathfrak{R}(\mathscr{G})$ into irreducible submodules.

Note that many results stated below about the integrability of the almost complex structure J_Z are known.

1. $G=\mathrm{GL}^+_{2n}(\mathbb{R})$

$$S=\mathrm{GL}^+_{2n}(\mathbb{R})/\mathrm{GL}_n(\mathbb{C}) ,$$

$$G(S)=\tilde{G}(S)=\mathrm{SL}_{2n}(\mathbb{R}) ;$$

$$\mathfrak{R}_+(m)\approx\mathfrak{R}_{\mathbb{C}}(\mathrm{gl}_n(\mathbb{C}))\neq\{0\} ,$$

$$\begin{aligned}\mathfrak{R}(\mathscr{G})&=d(\mathscr{G}^{(1)}\otimes V^*)=V\otimes S^2V^*\wedge V^*\\&=\mathfrak{R}_0(\mathscr{G})\oplus\mathfrak{R}_{S^2V^*}\oplus\mathfrak{R}_{\wedge^2V^*},\quad \mathfrak{R}_0(\mathscr{G})=\operatorname{Ker}\mathrm{ric} .\end{aligned}$$

The spaces $\mathfrak{R}_{\wedge^2V^*}$ and $\mathfrak{R}_{S^2V^*}$ consist of tensors of the form R_b,

$$R_b(x,y)=[b(x,y)-b(y,x)]E+x\otimes b_y-y\otimes b_x , \quad x, y\in V ,$$

where b is a skew-symmetric and symmetric bilinear form, respectively, and $b_y = b(y, \cdot)$.

Note that $\mathrm{ric}(R_b) = 2nb - b^t$ and the Weyl tensor of the curvature tensor R is called the projective curvature tensor.

(i) $\Leftrightarrow W^\omega \equiv 0$, that is, the connection ω is projectively flat [11].

(h) $\Leftrightarrow R^\omega \equiv 0$, that is, the connection ω is flat.

2. $G = \mathrm{Sp}_n(\mathbb{R})$, $V = \mathbb{R}^{2n}$

$$S = \mathrm{Sp}_n(\mathbb{R})/\mathrm{U}_n\,, \qquad G(S) = \tilde{G}(S) = \mathrm{Sp}_n(\mathbb{R})\,;$$

$$\mathfrak{R}_+(m) \neq 0\,,$$

$$\mathfrak{R}(\mathscr{G}) = \mathfrak{R}_0(\mathscr{G}) \oplus \mathfrak{R}_{S^2V^*} \approx \wedge^2 \mathscr{G}^*\,, \quad \mathfrak{R}_0(\mathscr{G}) = \mathrm{Ker\,ric}\,,$$

$$\mathfrak{R}_{S^2V^*} = \{R_a,\, a \in S^2V^*;\, \rho \circ R_a(x, y) = 2\rho(x, y)a - a_x \vee \rho_y + a_y \vee \rho_x\}\,,$$

where ρ is the standard symplectic form of the space $V = \mathbb{R}^{2h}$.

(i) $\Leftrightarrow W^\omega \equiv 0$,

(h) $\Leftrightarrow R^\omega \equiv 0$, that is, the connection ω is flat.

Remark. It is an interesting problem to construct and to study a manifold with a symplectic torsionless connection that has a vanishing Weyl tensor.

3. $G = \mathrm{SO}_{2n}$ *or* $\mathrm{CO}_{2n} = \mathbb{R}^+ \cdot \mathrm{SO}_{2n}$, $n \geqslant 2$

$$S = \mathrm{SO}_{2n}/\mathrm{U}_n \simeq \mathrm{CO}_{2n}/\mathbb{R}^+ \cdot \mathrm{U}_n\,,$$

$$G(S) = \tilde{G}(S) = \begin{cases} \mathrm{SO}_{2n}\,, & n > 2\,, \\ \mathrm{Sp}_1\,, & n = 2\,. \end{cases}$$

$$\mathfrak{R}_+(m) \neq 0\,,$$

$$\mathfrak{R}_-(f) = \begin{cases} 0\,, & \text{for } G = \mathrm{SO}_{2n},\, n \geqslant 2\,, \\ 0\,, & \text{for } G = \mathrm{CO}_{2n},\, n \geqslant 3\,, \\ \mathbb{C}\,, & \text{for } G = \mathrm{CO}_4,\, \mathbb{C} = \mathbb{R}\,\mathrm{id} + \mathbb{R}J_0\,. \end{cases}$$

(i) $\Leftrightarrow G(S) \cdot W^\omega = W^\omega$, that is, the Weyl tensor W^ω is invariant under the group of transvections,

(h) $\Leftrightarrow G(S) \cdot R^\omega = R^\omega$, that is, the curvature tensor is invariant under the group of transvections.

As a corollary, we have

Theorem 8.1. *Let M^{2n} be a manifold with a conformal structure and a torsionless conformal connection ω. Then*

(1) The almost complex structure J_Z in the associated twistor space Z is integra-

ble iff
– for $n>2$ the conformal structure is flat (that is, its Weyl tensor $W\equiv 0$), and
– for $n=2$ the conformal structure is anti-self-dual (that is, the self-dual part of the Weyl tensor $W_+\equiv 0$) [2,11];

(2) the canonical distribution $\mathscr{K}_Z$ is almost holomorphic iff
– for $n>2$ the connection ω is the Levi-Civita connection of a metric of constant curvature, and
– for $n=2$ the connection ω is anti-self-dual and its Ricci tensor ric satisfies the following conditions:

$$*\mathrm{ric}^{\mathrm{a}}=-\mathrm{ric}^{\mathrm{a}}\,,\qquad \mathrm{ric}^{\mathrm{s}}=fg\,,\qquad f\in C^{\infty}(M)\,,$$

where $\mathrm{ric}^{\mathrm{a}}$ *and* $\mathrm{ric}^{\mathrm{s}}$ *are the skew-symmetric and the symmetric parts of the tensor ric and g is a metric that defines the conformal structure.*

Example 8.2. The sphere $S^{2n}=SO_{2n+1}/SO_{2n}$ and the Lobachevsky space $\Lambda^{2n}=SO_{2n,1}/SO_{2n}$ have the twistor spaces

$$Z(S^{2n})=SO_{2n+1}/U_n\,,\qquad Z(\Lambda^{2n})=SO_{2n,1}/U_n\,,$$

with the integrable complex structure and the holomorphic canonical distributions.

In particular, the twistor space of the four-sphere is

$$Z(S^4)=SO_5/U_2\approx\mathbb{C}P^3\,,$$

and the holomorphic canonical distribution defines a SO_5-invariant complex contact structure in $\mathbb{C}P^3$.

Consider now the case when a linear group G is not simple and can be decomposed into the tensor product

$$G=G_1\otimes G_2\subset GL(V_1\otimes V_2)$$

of two linear groups $G_i\subset GL(V_i)$, $i=1, 2$. Assume that G_1 is a group of twistor type and that $J_1\in\mathscr{G}_1=\mathrm{Lie}\,G_1$ is a complex structure in the space V_1. Then $J_0=J_1\otimes 1\in\mathscr{G}_1\otimes 1+1\otimes\mathscr{G}_2=\mathscr{G}=\mathrm{Lie}\,G$ is a complex structure in the space $V=V_1\otimes V_2$. Hence, G is a group of twistor type.

Theorem 8.3. *Let $G=G_1\otimes G_2\subset GL(V_1\otimes V_2)$ be a group of twistor type and let $J_0=J_1\otimes 1$ be a complex structure in $V=V_1\otimes V_2$ that belongs to the ideal $\mathscr{G}_1\otimes 1$ of the Lie algebra $\mathscr{G}$. Let Z be the twistor space of a G-structure with torsionless connection ω, associated with J_0. Assume that* $\dim V_2>2$ *or* $\dim V_2=2$ *and the second prolongation of the algebra $\mathscr{G}_1\subset \mathrm{gl}(V_1)$ is equal to zero. Then condition (i) is satisfied and condition (h) is equivalent to the condition*

$$G(S)R^{\omega}=R^{\omega}\,.$$

The proof follows from theorems 7.8 and 7.5, if we observe that

$$m=[J_0,\mathscr{G}]\subset\mathscr{G}_1\,,\qquad f\cap J_0 f\subset\mathscr{G}_1\,,$$

and use the following

Lemma 8.4.

$$\mathfrak{R}(\mathscr{G}_1\otimes 1)\simeq\begin{cases}0\,, & \text{if } q=\dim V_2>2\,,\\ \mathscr{G}_1^{(2)}\,, & \text{if } q=2\,.\end{cases}$$

We note also that, in the case $q=2$, the space $\nabla\mathfrak{R}(\mathscr{G}_1\otimes 1)$ of first covariant derivatives of the curvature tensor is isomorphic to the space $\mathscr{G}_1^{(3)}\oplus\mathscr{G}_1^{(3)}$:

$$\nabla\mathfrak{R}(\mathscr{G}_1\otimes 1)\simeq\mathscr{G}_1^{(3)}\oplus\mathscr{G}_1^{(3)}\,.$$

We apply this theorem (more precisely, its complexified version) to the following two examples.

4. $G=\mathrm{Sp}_1\cdot\mathrm{Sp}_m$ *or* $\mathrm{Sp}_1\cdot\mathrm{GL}_m(\mathbb{H})$

$$S=\mathrm{Sp}_1/\mathrm{T}^1=\mathbb{C}\mathrm{P}^1\,,\qquad G(S)=\tilde{G}(S)=\mathrm{Sp}_1\,.$$

We shall assume that $m>1$, since for $m=1$ we have $\mathrm{Sp}_1\cdot\mathrm{Sp}_1=\mathrm{SO}_4$, $\mathrm{Sp}_1\cdot\mathrm{GL}_1(\mathbb{H})=\mathrm{CO}_4$, and these groups were considered before.

Condition (i) is fulfilled.

(h)$\Leftrightarrow\mathrm{Sp}_1\cdot R^\omega=R^\omega$.

This condition is always satisfied if $G=\mathrm{Sp}_1\cdot\mathrm{Sp}_m$. Hence, we have

Theorem 8.5 [3,4]. *The twistor space Z of a quaternionic Kähler or quaternionic manifold M^{4m}, $m>1$, has the integrable complex structure. The canonical distribution $\mathscr{H}_Z$ on Z is holomorphic if and only if the curvature tensor at each point is invariant under the group* Sp_1.

5. $G=\mathbb{R}^+\cdot\mathrm{SL}_p(\mathbb{H})\cdot\mathrm{SL}_q(\mathbb{H}),\ p>1,\ q>1$

Here $V=\mathbb{R}^{4pq}$, $J_0=\mathrm{diag}(\mathrm{i}E_p,\ -\mathrm{i}E_p)\in\mathrm{sl}_p(\mathbb{H})$, where E_p is the identity matrix of the order p. We have

$$S=\mathbb{R}^+\cdot\mathrm{SL}_p(\mathbb{H})/\mathrm{GL}_p(\mathbb{C})\,,\qquad G(S)=\tilde{G}(S)=\mathrm{SL}_p(\mathbb{H})\,.$$

We may apply the complexified version of theorem 8.3. It shows that condition (i) is fulfilled and condition (h) is equivalent to the condition

$$G(S)\cdot R^\omega=R^\omega\,.$$

Since any $G(S)$-invariant $gl_q(\mathbb{H})$-valued two-form on V is zero, the $G(S)$-invariant tensor R^ω lies in the space $\mathfrak{R}(sl_p(\mathbb{H}))$, which is zero by lemma 8.4. We obtain

Theorem 8.6. *Let $G=\mathbb{R}^+\cdot SL_p(\mathbb{H})\cdot SL_q(\mathbb{H})$, $p>1$, $q>1$. Then the twistor space Z of any G-structure with a torsionless connection has integrable complex structure J_Z. The canonical distribution $\mathcal{H}_Z$ is holomorphic if and only if the connection is flat.*

9. Case of connection with torsion

9.1. Let $\pi: P\to M$ be a G-structure of twistor type.

Denote by $\mathscr{C}$ the affine space of connections in π and by $\mathscr{J}$ ($\mathscr{J}_{\text{int}}$) the space of almost complex (complex) structures on the twistor space $Z=P/H$, see section 5. In section 5 we define a mapping

$$j: \mathscr{C}\to\mathscr{J}$$

that associates with a connection $\omega\in\mathscr{C}$ some almost complex structure $J^\omega=J_Z^\omega$ in Z. Now we describe a fiber of j.

Note that the difference $\gamma=\tilde{\omega}-\omega$ of two connections ω, $\tilde{\omega}\in\mathscr{C}$, is a G-equivariant horizontal $\mathfrak{g}$-valued one-form on P. It may be written as

$$\gamma=\Gamma\circ\theta=\Gamma_i\theta^i\,,$$

where $\Gamma=(\Gamma_1, ..., \Gamma_{2n}): \mathrm{P}\to C^1(\mathfrak{g})$ is a G-equivariant function and $\theta=(\theta^1, ..., \theta^{2n})$ is the displacement form.

As in section 6, we define the natural projections

$$\Pi^{p,q}_{1,0}: C^1(\mathfrak{g})\to C^{p,q}(\mathfrak{f})\,,$$
$$\Pi^{p,q}_{0,1}: C^1(\mathfrak{g})\to C^{p,q}(\mathfrak{m})\,,\quad p+q=1, p\geqslant 0, q\geqslant 0\,.$$

Theorem 9.1. *Let ω, $\tilde{\omega}\in\mathscr{C}$ and*

$$\tilde{\omega}-\omega=\gamma=\Gamma\circ\theta\,.$$

The connections ω, $\tilde{\omega}$ define the same almost complex structure $J^\omega=J^{\tilde{\omega}}$ iff $\Pi^{0,1}_{0,1}\Gamma_p=0$ $\forall p\in P$. Moreover, if $\Pi^{0,1}_{0,1}\Gamma_p=0$, then the almost complex structures J^ω, $J^{\tilde{\omega}}$ coincide in the point $z=\pi' p\in Z$.

Proof. We begin with some algebraic considerations. Let

$$(i,j)\quad 0\to Y\xrightarrow{i}T\xrightarrow{j}X\to 0$$

be a short exact sequence of real vector spaces, and let the short exact sequence

$$(\alpha, \beta) \quad 0 \leftarrow Y \overset{\beta}{\leftarrow} T \overset{\alpha}{\leftarrow} X \leftarrow 0$$

be its splitting, that is, $\beta \circ i = \mathrm{id}_Y$, $j \circ \alpha = \mathrm{id}_X$. The monomorphisms $i: Y \hookrightarrow T$, $j^*: X^* \hookrightarrow T^*$ induce monomorphisms $\mathrm{Hom}(X, Y) \hookrightarrow \mathrm{Hom}(X, T)$, $\mathrm{Hom}(X, Y) \hookrightarrow \mathrm{Hom}(T, Y)$. It defines an action of the vector group $\mathrm{Hom}(X, Y)$ on the set of splittings (α, β) by

$$\mathrm{Hom}(X, Y) \ni \varphi: (\alpha, \beta) \mapsto (\alpha - \varphi, \beta + \varphi) .$$

Assume that complex structures J_X, J_Y in the spaces X, Y are given. Then we have the decomposition

$$\mathrm{Hom}_{\mathbb{R}}(X, Y) = \mathrm{Hom}^{1,0}(X, Y) + \mathrm{Hom}^{0,1}(X, Y) ,$$

where $\mathrm{Hom}^{1,0}$ ($\mathrm{Hom}^{0,1}$) is the space of holomorphic (antiholomorphic) homomorphisms.

A splitting (α, β) of the sequence (i, j) determines a unique complex structure $J(\alpha, \beta)$ in T such that (i, j) is a sequence of complex vector spaces. We have the following algebraic

Lemma 9.2.

(1) Two splittings (α, β), $(\tilde{\alpha}, \tilde{\beta})$ of an exact sequence (i, j) define the same complex structure $J(\alpha, \beta) = J(\tilde{\alpha}, \tilde{\beta})$ in T iff

$$\alpha - \tilde{\alpha} = \tilde{\beta} - \beta \in \mathrm{Hom}^{1,0}(X, Y) .$$

(2) The group $\mathrm{Hom}^{0,1}(X, Y)$ acts simply transitively on the set of complex structures of the form $J(\alpha, \beta)$ as follows:

$$\mathrm{Hom}^{0,1}(X, Y) \ni \varphi: J(\alpha, \beta) \mapsto J(\alpha - \varphi, \beta + \varphi) .$$

To prove theorem 9.1, we apply the lemma to two complex structures J^{ω}, $J^{\bar{\omega}}$ of the tangent space $T_z Z$ of a point $z \in Z$. We have the short exact sequence

$$(i, j) \quad 0 \to T_z^v Z \overset{i}{\to} T_z Z \xrightarrow{j = \tau_*} T_x M \to 0 , \quad x = \pi'(z) .$$

A point $p \in \pi^{-1}(x)$ determines two isomorphisms $v_p: T_z^v Z \to \mathfrak{m}$ and $\theta_p: T_x M \to V$. Hence, the complex structures $J_{\mathfrak{m}}$, J_V induce complex structures in $T_z^v Z$ and $T_x M$.

Each connection ω defines a splitting (α, β) of the sequence (i, j):

$$\mathrm{Im}\, \alpha = \mathrm{Ker}\, \beta = \mathscr{H}_Z .$$

We have the following formula:

$$v_p \circ \beta_z \circ \pi'_{*p} = \omega_p^{\mathfrak{m}} . \qquad (*)$$

It is easy to check that

$$J_z^{\omega} = J(\alpha, \beta) .$$

Let $\tilde{\omega} = \omega + \Gamma \circ \theta = \omega + \gamma$ be another connection. It defines a new splitting $(\tilde{\alpha}, \tilde{\beta})$ and a new complex structure

$$J_z^{\tilde{\omega}} = J(\tilde{\alpha}, \tilde{\beta}) .$$

According to lemma 9.2,

$$J_z^{\omega} = J_z^{\tilde{\omega}} \Leftrightarrow \alpha - \tilde{\alpha} = \tilde{\beta} - \beta \in \mathrm{Hom}^{1,0}(T_x M, T_z^v Z) .$$

We may rewrite the last condition as follows:

$$v_p \circ (\tilde{\beta} - \beta) \circ \theta_p^{-1} \in \mathrm{Hom}^{1,0}(V, \mathfrak{m}) .$$

Using formula ($*$), the last expression may be written as

$$\begin{aligned} v_p \circ (\tilde{\beta}_z - \beta_z) \circ \theta_p^{-1} &= v_p \circ (\tilde{\beta}_z - \beta_z) \circ \pi'_{*p} \circ \theta_p^{-1} \\ &= (\tilde{\omega}_p - \omega_p^{\mathfrak{m}}) \circ \theta_p^{-1} = \gamma_p^{\mathfrak{m}} \circ \theta_p^{-1} = \Gamma_p^{\mathfrak{m}} . \end{aligned}$$

Hence, the equality $J_p^{\omega} = J_p^{\tilde{\omega}}$ holds iff $\Gamma_p^{\mathfrak{m}} \equiv \Pi_{0,1} \Gamma_p \in \mathrm{Hom}^{1,0}(V, \mathfrak{m}) := C^{1,0}(\mathfrak{m})$ or, in other terms, iff

$$\Pi_{0,1}^{0,1} \Gamma_p = 0 \quad \forall p \in P .$$

This proves theorem 9.1. □

9.2. We define two equivalence relations $\overset{t}{\sim}$ and $\approx$ in $\mathscr{C}$.

Definition 9.3. Let $\omega, \tilde{\omega} \in \mathscr{C}$ be two connections with the torsion forms $\Theta, \tilde{\Theta}$. The connections ω, $\tilde{\omega}$ are called Tor-equivalent ($\mathrm{Tor}^{0,2}$-equivalent) iff $\Theta = \tilde{\Theta}$ ($\Theta^{0,2} = \tilde{\Theta}^{0,2}$), where $\Theta^{0,2}$ is the $(0, 2)$-component of Θ. In this case we shall write $\omega \overset{t}{\sim} \tilde{\omega}$ $(\omega \approx \omega)$.

It is clear that $\omega \overset{t}{\sim} \tilde{\omega}$ implies $\omega \approx \tilde{\omega}$.

Lemma 9.4. *Let $\tilde{\omega} = \omega + \Gamma \circ \theta$. Then*

$$\begin{aligned} \tilde{\omega} \overset{t}{\sim} \omega \quad &\Leftrightarrow \quad d\Gamma = 0 , \\ \tilde{\omega} \approx \omega \quad &\Leftrightarrow \quad \Pi_{0,0}^{0,2} d\Gamma = d\Pi_{0,1}^{0,1} \Gamma = 0 \\ &\Leftrightarrow \quad \Gamma_p \in \mathfrak{g}^{(1)} + C^1(\mathfrak{g}) \cap \mathrm{Ker}\, \Pi_{0,1}^{0,1} \quad \forall p \in P , \end{aligned}$$

where

$$d: C_1^1 = V \otimes V^* \otimes V^* \to C_0^2 = V \otimes \wedge^2 V^*$$

is the Koszul–Spencer differential.

Theorem 9.5. *Let* $\omega_0 \in \mathscr{C}$.

(1) Then the sets $\{J^\omega, \omega \overset{1}{\sim} \omega_0\}$, $\{J^\omega, \omega \approx \omega_0\}$ *consist of one element iff*

$$0 = \mathfrak{m}^{(1)} := (\operatorname{Ker} d) \cap C^1(\mathfrak{m}) = \Pi^{0,1}_{0,1}\mathfrak{g}^{(1)} .$$

(2) Assume that the group G is reductive (or more generally, $C^1(\mathfrak{g}) = \mathfrak{g}^{(1)} \oplus \mathscr{D}$, *where* $\mathscr{D}$ *is a* $\tilde{G}(S)$*-invariant complement to the first prolongation* $\mathfrak{g}^{(1)}$ *of* $\mathfrak{g}$*). Then*

$$\{J^\omega, \omega \overset{1}{\sim} \omega_0\} = \{J^\omega, \omega \approx \omega_0\} .$$

Corollary 9.6.
(1) Assume that $\mathfrak{g}^{(2)} = 0$ *and* $\mathfrak{g}^{(1)} = V^*$. *Then* $\mathfrak{m}^{(1)} = 0$ *and two connections* ω, $\tilde{\omega}$ *with the torsion forms* Θ, $\tilde{\Theta}$ *define the same almost complex structure iff*

$$\Theta^{0,2} = \tilde{\Theta}^{0,2} .$$

(2) Assume that G is a reductive group and ω_0 *is a torsionless connection. Then any almost complex structure J from the set* $\{J^\omega, \omega \approx \omega_0\}$ *is determined by a* torsionless *connection* ω. *The converse statement is also true.*

Remark 9.7. This corollary may be applied to an irreducible linear group of type 2, in particular to

$$G = \mathrm{CO}_n , \quad \mathrm{GL}_1(\mathbb{H}) \cdot \mathrm{Sp}_m , \quad \mathrm{GL}_p(F) \cdot \mathrm{GL}_q(F) , \quad F = \mathbb{R}, \mathbb{C} .$$

9.3. Now we characterize a connection from the set

$$\mathscr{C}_{\text{int}} = j^{-1}(\mathscr{J}_{\text{int}} \cap j(\mathscr{C})) ,$$

that is, a connection ω that determines an integrable complex structure J^ω. Recall (see section 7) that the curvature form of a connection ω defines the curvature function

$$R^\omega : P \to C^2(\mathfrak{g}) .$$

In the same way, the torsion form Θ defines the torsion function

$$T^\omega : P \to C^2_0 = V \otimes \wedge^2 V^* ,$$

where

$$T^\omega_p = \Theta_p(\hat{p}^{-1} \cdot, \hat{p}^{-1} \cdot) ,$$

and $\hat{p} : T_{\pi p} M \to V$ is the coframe associated with a point $p \in P$.

Theorem 9.8. *The almost complex structure J^ω associated with a connection $\omega\in\mathscr{C}$ is integrable iff*

$$\Pi^{0,2}T_p^\omega=\Pi_{0,1}^{0,2}R_p^\omega=0 \quad \forall p\in P.$$

This theorem is an obvious reformulation of theorem 5.2 that gives the integrability conditions of J^ω in the form

$$\Theta^{0,2}=0, \qquad (\Omega^{\mathfrak{m}})^{0,2}=0.$$

Corollary 9.9.

$$\mathscr{C}_{\text{int}}=\{\omega\in\mathscr{C}, \Pi^{0,2}T^\omega=\Pi_{0,1}^{0,2}R^\omega=0\}.$$

In particular, the set $\mathscr{C}_{\text{int}}$ is contained in the affine space

$$\mathscr{C}_0=\{\omega\in\mathscr{C}, \Pi^{0,2}T^\omega=0\}.$$

Proposition 9.10. *Assume that $\mathscr{C}_0\neq\emptyset$ and $\omega_0\in\mathscr{C}_0$. Then*

(1) $\mathscr{C}_0=\{\omega\in\mathscr{C}, \omega\approx\omega_0\}=j^{-1}(\mathscr{J}_0)$,
where $\mathscr{J}_0$ is some subset of $\mathscr{J}$.

(2) If $\mathfrak{m}^{(1)}=0$, then $\mathscr{J}_0$ is a point.

(3) If G is a reductive group, then

$$\mathscr{J}_0=\{J^\omega, \omega\in\mathscr{C}, T^\omega=T^{\omega_0}\}.$$

In particular, if there exists a torsionless connection ω_0, then

$$\mathscr{J}_0=\{J^\omega, \omega\in\mathscr{C}, T^\omega=0\}.$$

Proof.
(1) Follows from theorem 9.1 and definition.
(2) Follows from theorem 9.5(1).
(3) Follows from theorem 9.5(2).

9.4. By theorem 9.8, the almost complex structure J^ω associated with a connection $\omega\in\mathscr{C}_0$ is integrable iff the component $\Pi_{0,1}^{0,2}R^\omega$ of the curvature function $R^\omega:P\to C^2(\mathfrak{g})$ vanishes.

The following important theorem describes the algebraic properties of the function $\Pi_{0,1}^{0,2}R^\omega$, which is the obstacle for integrability of J^ω.

Theorem 9.11. *Let $\omega\in\mathscr{C}_0$. Then*

(1) the complex structure J^ω is integrable iff $\Pi_{0,1}^{0,2}R^\omega\equiv0$;

(2) the function $p\to(\Pi_{0,1}^{0,2}R^\omega)_p$ takes values in the space $\mathfrak{R}_+(\mathfrak{m})$,

$$\Pi^{0,2}_{0,1} R^{\omega}: P \to \mathfrak{R}_+(\mathfrak{m}) , \qquad (*)$$

and, in particular, it satisfies the algebraic Bianchi identity

$$d\Pi^{0,2}_{0,1} R^{\omega}_p = 0 \quad \forall p \in P .$$

Proof. It is only necessary to prove (2). It follows from the following lemmas.

Lemma 9.12. *For any* $\omega \in \mathscr{C}$ *we set*

$$B^{\omega} = R^{\omega} - \nabla T^{\omega} - T^{\omega} \circ T^{\omega} ,$$

where ∇T^{ω} *is the function associated with the covariant derivative of the torsion tensor and*

$$T^{\omega} \circ T^{\omega}(x, y) = T^{\omega}(T^{\omega}(x, y), \cdot)$$

for tangent vectors x, y.

Then the Bianchi identity may be written as

$$dB^{\omega} = 0 .$$

See Lichnerowicz [12].

Lemma 9.13. *If* $\omega \in \mathscr{C}_0$, *then*

$$\Pi^{0,2}_{0,1} B^{\omega} = \Pi^{0,2}_{0,1} R^{\omega} .$$

Proof. We must check that

$$\Pi^{0,2}(\nabla T^{\omega}) = 0 , \qquad \Pi^{0,2}_{0,1}(T^{\omega} \circ T^{\omega}) = 0 .$$

The first equality is derived from the condition $\Pi^{0,2} T^{\omega}_p = 0$, $\forall p \in P$. The second one follows from an analysis of eigenvalues of the operator $\hat{J}$ in the spaces $C^2_{\bar{0}}$ and $C^2_{\bar{1}}$, see section 6.

Using these lemmas and the identity $\Pi^{0,3}_{0,0} \circ d = d \circ \Pi^{0,2}_{0,1}$, we may write

$$d(\Pi^{0,2}_{0,1} R^{\omega}) = d(\Pi^{0,2}_{0,1} B^{\omega}) = \Pi^{0,3}_{0,0} dB^{\omega} = 0 .$$

The results of section 6 now imply that $\Pi^{0,2}_{0,1} R^{\omega}_0 \in \mathfrak{R}_+(\mathfrak{m})$. □

Remark 9.14.

(1) In general, $dR^{\omega} \neq 0$.

(2) By virtue of (*) the results of section 6 about the integrability conditions for J^{ω} are generalized to the case of a connection with torsion. We will not state it here.

(3) A simpler proof of theorem 9.11 based on Cartan's connections will be published elsewhere.

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